CALCULUS BC
WORKSHEET 3 ON SERIES

Work all problems on separate paper. You may use your calculator on problems 2 and 4 only.

1. Let g be the function given by $g(x) = \dfrac{\sin x}{x}$.

(a) Write the first four nonzero terms and the general term for the series for $\sin x$ centered at $x = 0$.

(b) Use your results from part (a) to write the first four nonzero terms and the general term for the series for $g(x)$.

(c) Use the first two terms of your series in part (b) to estimate $g(1)$.

(d) Show that the approximation found in part (c) approximates $g(1)$ with error less than $\dfrac{1}{100}$.

2. The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n (n+2)}$ and $f(5) = \dfrac{1}{2}$. Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

3. Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

(a) Does f have a local maximum, local minimum, or neither at $x = 2$? Justify your answer.

(b) Use $T(x)$ to find an approximation for $f(0)$.

(c) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (b) to explain why $f(0)$ must be negative.

4. The function f has derivatives of all orders for all real numbers x. Assume that $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

(b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \ne -5$.

5. Let f be the function given by $f(x)=\sin\left(5x+\frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree polynomial for ` f about $x=0$. Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right)-P\left(\frac{1}{10}\right)\right|<\frac{1}{100}$.

6. The function f is defined by the power series

$$f(x)=\sum_{n=0}^{\infty}\frac{(x-2)^n}{3^n(n+1)}=1+\frac{x-2}{3\cdot 2}+\frac{(x-2)^2}{3^2\cdot 3}+\frac{(x-2)^3}{3^3\cdot 4}+\ldots+\frac{(x-2)^n}{3^n(n+1)}+\ldots$$

for all real numbers x for which the series converges.

(a) Find the value of $f''(2)$.

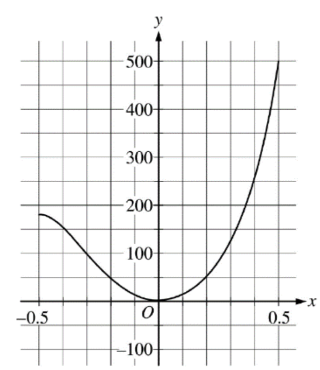
(b) Use the first three nonzero terms of the power series for f to approximate $f(1)$. Use the alternating series error bound to show that this approximation differs from $f(1)$ by less than $\frac{1}{100}$.

7. (Multiple Choice) The Taylor series for a function f about $x=0$ converges for $-1\le x\le 1$. The nth-degree Taylor polynomial for f about $x=0$ is given by $P_n(x)=\sum_{k=1}^{n}(-1)^k\frac{x^k}{k^2+k+1}$. Of the following, which is the smallest number M for which the alternating series error bound guarantees that $|f(1)-P_4(1)|\le M$?

(A) $\frac{1}{5!}\cdot\frac{1}{31}$ (B) $\frac{1}{4!}\cdot\frac{1}{21}$ (C) $\frac{1}{31}$ (D) $\frac{1}{21}$

8. The Taylor series for a function f about $x=0$ is given by $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}x^{2n}$ and converges to f for all real numbers x. If the fourth-degree Taylor polynomial for f about $x=0$ is used to approximate $f\left(\frac{1}{2}\right)$, what is the alternating series error bound?

9. Let f and g be the functions given by $f(x)=xe^{x^3}$ and $g(x)=\int_0^x f(t)\,dt$. The graph of $f^{(5)}$, the fifth derivative of f, is shown above for $-\frac{1}{2}\le x\le\frac{1}{2}$.



Graph of $f^{(5)}$

(a) Write the first four nonzero terms and the general term of the Taylor series for e^x about $x=0$.

Write the first four nonzero terms and the general term of the Taylor series for f about $x=0$.

(b) Write the first four nonzero terms of the Taylor series for g about $x=0$.

(c) Let $P_5(x)$ be the fifth-degree Taylor polynomial for g about $x=0$. Use the Lagrange error bound along with information from the given graph to find an upper bound on $\left|P_5\left(\frac{1}{2}\right)-g\left(\frac{1}{2}\right)\right|$.